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Iterative Approximation of Fixed Points of Generalized Weak Presic Type k-Step Iterative Method for a Class of Operators

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Abstract. In this paper, we study the convergence of the generalized weak Presic type k-step iterative method for a class of operators $f : X^k \to X$ satisfying Presic type contractive conditions. We also obtain the global attractivity results for a class of matrix difference equations.

1. Introduction and Preliminaries

Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular, the applications of fixed point theorems are very important in diverse disciplines of mathematics, economics, statistics and engineering in dealing with problems arising in: mathematical economics, game theory, approximation theory, potential theory, etc. Banach contraction principle [2] is simple and powerful result with a wide range of applications, including iterative methods for solving linear, nonlinear, differential, integral, and difference equations. There are several generalizations and extensions of the Banach contraction principle in the existing literature. Banach contraction principle reads as follows:

Theorem 1.1. [2] Let (X, d) be a complete metric space and mapping $f : X \to X$ satisfies

$$d(fx, fy) \leq \alpha\, d(x, y), \text{ for all } x, y \in X,$$

where $\alpha \in [0, 1)$ is a constant. Then there exists a unique $x \in X$ such that $x = fx$. Moreover, for any $x_0 \in X$, the iterative sequence $x_{n+1} = f(x_n)$ converges to x.

Definition 1.2. A mapping $f : X \to X$ is said to be a weakly contractive if

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)), \text{ for all } x, y \in X,$$

where $\varphi : [0, \infty) \to: [0, \infty)$ is a continuous and non-decreasing function such that it is positive in $(0, \infty)$, $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

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In 1997, Alber and Guerre-Delabriere [1] proved that weakly contractive mapping defined on a Hilbert space is a Picard operator. Rhoades [19] proved that the corresponding result is also valid when Hilbert space is replaced by a complete metric space. Dutta et al. [10] generalized the weak contractive condition and proved a fixed point theorem for a selfmap, which in turn generalizes Theorem 1 in [19] and the corresponding result in [1].

Let $f : X^k \to X$, where $k \geq 1$ is a positive integer. A point $x^* \in X$ is called a fixed point of f if $f(x^*, \dots, x^*) = x^*$. Consider the k-th order nonlinear difference equation

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}),\ n = 1, 2, \dots \tag{1.1}$$

with the initial values $x_1, \dots, x_k \in X$.
Equation (1.1) can be studied by means of fixed point theory in view of the fact that x in X is a solution of (1.1) if and only if x is a fixed point of the self-mapping $F : X \to X$ given by

$$F(x) = f(x, \dots, x),\ \text{for all } x \in X.$$

Prešić [18] obtained the following result in this direction.
Theorem 1.3. [18] Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a mapping satisfying the following contractive type condition

$$d(f(x_1, x_2, \dots, x_k), f(x_2, \dots, x_k, x_{k+1})) \leq a_1 d(x_1, x_2) + a_2 d(x_2, x_3) + \dots + a_k d(x_k, x_{k+1}),$$

for every $x_1, \dots, x_{k+1} \in X$, where $a_1, a_2, \dots, a_k \geq 0$ with $q_1 + q_2 + \dots + q_k < 1$. Then there exists a unique point $x^* \in X$ such that $f(x^*, \dots, x^*) = x^*$. Moreover, for any arbitrary points $x_1, \dots, x_k \in X$, the sequence (1.1) converges to x^*.

If we take $k = 1$, then Theorem 1.2 reduces to the Banach contraction principle.

Ćirić and Prešić [8] generalized the above theorem as follows.
Theorem 1.4. [8] Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a mapping satisfying the following contractive type condition

$$d(f(x_1, x_2, \dots, x_k), f(x_2, \dots, x_k, x_{k+1})) \leq h \max\{d(x_1, x_2), d(x_2, x_3), \dots, d(x_k, x_{k+1})\},$$

for every $x_1, \dots, x_{k+1} \in X$, where $0 < h < 1$ is a constant. Then there exists $x^* \in X$ such that $f(x^*, \dots, x^*) = x^*$. Moreover, for any arbitrary points $x_1, \dots, x_k \in X$, the sequence (1.1) is convergent and

$$\lim_{n\to\infty} x_n = f(\lim_{n\to\infty} x_n, \dots, \lim_{n\to\infty} x_n).$$

Furthermore, we suppose that

$$d(T(u, \dots, u), T(v, \dots, v)) < d(u, v)$$

holds for all $u, v \in X$, with $u \neq v$, then x^* is the unique point in X with $f(x^*, \dots, x^*) = x^*$.

The applicability of the above result to the study of global asymptotic stability of the equilibrium for the nonlinear difference equation (1.1) is revealed, for example, see [6].

In [17], Păcurar obtained the following convergence result for Prešić-Kannan operators.
Theorem 1.5. [17] Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a given mapping. Suppose that there exists a constant $a \in \mathbb{R}$ with $0 < ak(k+1) < 1$ such that

$$d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1})) \leq a \sum_{i=1}^{k+1} d(x_i, f(x_i, \dots, x_i)),$$

for all $(x_1, \dots, x_{k+1}) \in X^{k+1}$. Then,
(i) f has a unique fixed point $x^* \in X$;

(ii) for any arbitrary points $x_1, \dots, x_k \in X$, the sequence $\{x_n\}$ defined by (1.1) converges to x^*.

For other results on Prešić operators, we refer to [3, 4, 6, 8, 13, 16, 17, 20–24].

In this paper, we study the convergence of the sequence $\{x_n\}$ defined by (1.1) for the mapping $f : X^k \to X$ satisfies various contractive conditions of Prešić type. We also present an example and application of obtained result.

We denote by $\mathbb{R}$ the set of all real numbers, $\mathbb{R}^+$ the set of all nonnegative real numbers and $\mathbb{N}$ the set of all positive integers.

2. Main Results

We start with the following result.

Theorem 2.1. Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a given mapping. Suppose that there exists $\phi : [0, \infty) \to [0, \infty)$ a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$ such that

$$d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1})) \leq \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\} - \phi(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}), \tag{2.1}$$

for all $(x_1, \dots, x_{k+1}) \in X^{k+1}$. Then, for any arbitrary points $x_1, \dots, x_k \in X$, the sequence $\{x_n\}$ defined by (1.1) converges to $u \in X$ and u is a fixed point of f, that is, $u = f(u, \dots, u)$. Moreover, if

$$d(f(x, \dots, x), f(y, \dots, y)) \leq d(x, y) - \phi(d(x, y)), \tag{2.2}$$

holds for all $x, y \in X$ with $x \neq y$, then u is the unique fixed point of f.

Proof. Let $x_1, \cdots, x_k$ be arbitrary k elements in X. Define the sequence $\{x_n\}$ in X by

$$x_{n+k} = f(x_n, x_{n+1}, ..., x_{n+k-1}), \ n = 1, 2, \dots.$$

For $n \leq k$, by using (2.1), we have the following inequalities:

$$\begin{aligned} d(x_{k+1}, x_{k+2}) &= d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1})) \\ &\leq \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\} - \phi\left(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}\right) \\ &< \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}, \end{aligned}$$

$$\begin{aligned} d(x_k, x_{k+1}) &= d(f(x_1, \dots, x_{k-1}), f(x_2, \dots, x_k)) \\ &\leq \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k-1\} - \phi\left(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k-1\}\right) \\ &< \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k-1\}, \\ &\ \ \vdots \end{aligned}$$

$$\begin{aligned} d(x_{k-n}, x_{k-n+1}) &= d(f(x_1, \dots, x_{k-n-1}), f(x_2, \dots, x_{k-n})) \\ &\leq \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k-n-1\} - \phi\left(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k-n-1\}\right) \\ &< \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k-n-1\}. \end{aligned}$$

We conclude that $\{d(x_{n+k-1}, x_{n+k})\}$ is monotone nonincreasing and bounded below. So there exists some $c \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+k-1}, x_{n+k}) = \lim_{n\to\infty} \max\{d(x_{n+i}, x_{n+i+1}) : 1 \leq i \leq k-1\} = c.$$

We claim that $c = 0$. In fact, taking upper limits as $n \to \infty$ on either side of the following inequality:

$$\begin{aligned} d(x_{k+n}, x_{k+n+1}) &= d(f(x_1, \dots, x_{k+n-1}), f(x_2, \dots, x_{k+n})) \\ &\leq \max\{d(x_{i+n}, x_{i+n+1}) : 1 \leq i \leq k-1\} - \phi\left(\max\{d(x_{i+n}, x_{i+n+1}) : 1 \leq i \leq k-1\}\right), \end{aligned}$$

we have

$$c \leq c - \phi(c),$$

that is, $\phi(c) \leq 0$. Thus $\phi(c) = 0$ by the property of ϕ, and furthermore

$$\lim_{n\to\infty} d(x_{n+k-1}, x_{n+k}) = 0. \tag{2.3}$$

Next we show that $\{x_n\}$ is Cauchy. For any $n, m \in \mathbb{N}$ with $m \geq n$, using (2.1) we have

$$\begin{aligned}
d(x_{k+n}, x_{k+m}) &= d(f(x_1, \ldots, x_{k+n-1}), f(x_2, \ldots, x_{k+m-1})) \\
&\leq d(f(x_1, \ldots, x_{k+n-1}), f(x_2, \ldots, x_{k+n})) + d(f(x_2, \ldots, x_{k+n}), f(x_3, \ldots, x_{k+n+1})) \\
&\quad +\ldots + d(f(x_2, \ldots, x_{k+n}), f(x_3, \ldots, x_{k+m-1})) \\
&\leq \max\{d(x_{i+n}, x_{i+n+1}) : 1 \leq i \leq k-1\} - \phi(\max\{d(x_{i+n}, x_{i+n+1}) : 1 \leq i \leq k-1\}) \\
&\quad + \max\{d(x_{i+n}, x_{i+n+1}) : 1 \leq i \leq k\} - \phi(\max\{d(x_{i+n}, x_{i+n+1}) : 1 \leq i \leq k\}) \\
&\quad +\ldots + \max\{d(x_{i+n}, x_{i+n+1}) : 1 \leq i \leq k+m-1\} \\
&\quad -\phi(\max\{d(x_{i+n}, x_{i+n+1}) : 1 \leq i \leq k+m-1\}).
\end{aligned}$$

On taking the upper limit as $n, m \to \infty$ implies that

$$\lim_{n\to\infty} d(x_{k+n}, x_{k+m}) = 0.$$

Hence $\{x_n\}$ is also a Cauchy sequence in (X, d). Since (X, d) is complete, there exists u in X such that

$$\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n\to\infty} d(x_n, u). \tag{2.4}$$

Now, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(u, f(u, u, \ldots, u)) &\leq d(u, x_{n+k}) + d(x_{n+k}, f(u, u, \ldots, u)) \\
&\leq d(u, x_{n+k}) + d(f(x_n, x_{n+1}, \ldots, x_{n+k-1}), f(u, u, \ldots, u)) \\
&\leq d(u, x_{n+k}) + d(f(u, u, \ldots, u), f(u, u, \ldots, x_n)) \\
&\quad +d(f(u, u, \ldots, x_n), f(u, \ldots, x_n, x_{n+1})) \\
&\quad + \ldots + d(f(u, x_n, x_{n+1}, \ldots, x_{n+k-2}), f(x_n, x_{n+1}, \ldots, x_{n+k-1})) \\
&\leq d(u, x_{n+k}) + d(u, x_n) - \phi(d(u, x_n)) \\
&\quad + \max\{d(u, x_n), d(x_n, x_{n+1})\} - \phi(\max\{d(u, x_n), d(x_n, x_{n+1})\}) \\
&\quad + \ldots + \max\{d(u, x_n), d(x_n, x_{n+1}), \ldots, d(x_{n+k-2}, x_{n+k-1})\} \\
&\quad -\phi(\max\{d(u, x_n), d(x_n, x_{n+1}), \ldots, d(x_{n+k-2}, x_{n+k-1})\}).
\end{aligned}$$

On taking upper limit as $n \to \infty$ in the above inequality and using (2.4), we obtain

$$d(u, f(u, u, \ldots, u)) \leq 0,$$

which implies that $u = f(u, u, \ldots, u)$, that is, u is a fixed point of f.

To prove the uniqueness of the fixed point, assume that there exists an element $v \in X$ with $v \neq u$, such that $v = f(v, v, \ldots, v)$. Then by (2.2), we have

$$\begin{aligned}
d(u, v) &= d(f(u, u, \ldots, u), f(v, v, \ldots, v)) \\
&\leq d(u, v) - \phi(d(u, v)) \\
&< d(u, v),
\end{aligned}$$

a contradiction. So, u is the unique point in X such that $u = f(u, u, \ldots, u)$. □

Example 2.2. Let $X = [0, 2]$ and d be a usual metric of X. Let k be a positive integer and $f : X^k \to X$ be the mapping defined by

$$f(x_1, \ldots, x_k) = \frac{x_1 + \ldots + x_k}{4k} \text{ for all } x_1, \ldots, x_k \in X.$$

Define $\phi : [0, \infty) \to [0, \infty)$ by

$$\phi(t) = \begin{cases} \frac{t}{5}, & \text{if } t \in [0, \frac{5}{2}), \\ \frac{2^n(2^{n+1}t - 3)}{2^{2n+1} - 1}, & \text{if } t \in [\frac{2^{2n} + 1}{2^n}, \frac{2^{2(n+1)} + 1}{2^{n+1}}],\ n \in \mathbb{N}. \end{cases}$$

An easy computation shows that ϕ is lower semi-continuous on $[0, \infty)$ and $\phi(t) = 0$ if and only if $t = 0$. Now, for all $x_1, x_2, \ldots, x_{k+1} \in X$, we have

$$\begin{aligned} d(f(x_1, \ldots, x_k), f(x_2, \ldots, x_{k+1})) &= \frac{1}{4k} |x_1 - x_{k+1}| \\ &\leq \frac{1}{4} \max\{|x_i - x_{i+1}| : 1 \leq i \leq k\} \\ &\leq \frac{4}{5} \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\} \\ &= \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\} - \phi\left(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}\right). \end{aligned}$$

Moreover, for all $x, y \in X$ with $x \neq y$, the equation

$$d(f(x, \ldots, x), f(y, \ldots, y)) < d(x, y) - \phi\left(d(x, y)\right)$$

hold. Thus, all the required hypotheses of Theorem 2.1 are satisfied, we deduce that for any arbitrary points $x_1, ..., x_k \in X$, the sequence $\{x_n\}$ defined by (1.1) converges to $u = 0$, which is the unique fixed point of f. □

By taking $\phi(t) = (1 - \lambda)t$ for all $t \in [0, \infty)$ in Theorem 2.1, we obtain the following immediate consequence of Theorem 2.1.

Corollary 2.3. Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a given mapping. Suppose that there exists $\lambda \in [0, 1)$ such that

$$d(f(x_1, \ldots, x_k), f(x_2, \ldots, x_{k+1})) \leq \lambda \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}, \tag{2.5}$$

for all $(x_1, \ldots, x_{k+1}) \in X^{k+1}$. Then, for any arbitrary points $x_1, \ldots, x_k \in X$, the sequence $\{x_n\}$ defined by (1.1) converges to u and u is a fixed point of f, that is, $u = f(u, \ldots, u)$. Moreover, if

$$d(f(x, \ldots, x), f(y, \ldots, y)) \leq \lambda d(x, y),$$

holds for all $x, y \in X$ with $x \neq y$, then u is the unique fixed point of f.

Corollary 2.4. Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a given mapping. Suppose that there exist $\lambda_1, \ldots, \lambda_k$ non-negative constants with $\lambda_1 + \lambda_2 + \ldots + \lambda_k < 1$ such that

$$d(f(x_1, \ldots, x_k), f(x_2, \ldots, x_{k+1})) \leq \lambda_1 d(x_1, x_2) + \lambda_2 d(x_2, x_3) + \ldots + \lambda_k d(x_k, x_{k+1}), \tag{2.6}$$

for all $(x_1, \ldots, x_{k+1}) \in X^{k+1}$. Then, for any arbitrary points $x_1, ..., x_k \in X$, the sequence $\{x_n\}$ defined by (1.1) converges to u, where u is the unique fixed point of f.

Proof. Clearly, condition (2.6) implies condition (2.5) with $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_k$. Now, let $x, y \in X$ with $x \neq y$. From (2.6), we have

$$\begin{aligned} d(f(x, x, \ldots, x), f(y, y, \ldots, y)) &\leq d(f(x, \ldots, x), f(x, \ldots, x, y)) + d(f(x, \ldots, x, y), f(x, \ldots, x, y, y)) \\ &\quad + \ldots + d(f(x, y, \ldots, y), f(y, y, \ldots, y)) \\ &\leq (\lambda_k + \lambda_{k-1} + \ldots + \lambda_1) d(x, y) = \lambda d(x, y), \end{aligned}$$

where $\lambda = \lambda_k + \lambda_{k-1} + \ldots + \lambda_1 \in [0, 1)$. Finally, all the hypotheses of Corollary 2.3 are satisfied, then we deduce the desired result. □

Theorem 2.5. Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a given mapping. Suppose that there exists a constant $a \in \mathbb{R}$ with $0 \leq ak < 1$ such that

$$d(f(x_1, \ldots, x_k), f(x_2, \ldots, x_{k+1})) \leq a \max\{d(x_i, f(x_i, \ldots, x_i)) : 1 \leq i \leq k+1\}, \tag{2.7}$$

for all $(x_1, \ldots, x_{k+1}) \in X^{k+1}$. Then,
(i) f has a unique fixed point $u \in X$;
(ii) for any arbitrary points $x_1, \ldots, x_k \in X$, the sequence $\{x_n\}$ defined by (1.1) converges to u.
Proof. Define the mapping $F : X \to X$ by

$$F(x) = f(x, x, \ldots, x), \text{ for all } x \in X.$$

For all $x, y \in X$, we have

$$\begin{aligned} d(F(x), F(y)) &= d(f(x, x, \ldots, x), f(y, y, \ldots, y)) \\ &\leq d(f(x, x, \ldots, x), f(x, \ldots, x, y)) + d(f(x, \ldots, x, y), f(x, \ldots, x, y, y)) \\ &\quad + \ldots + d(f(x, y, \ldots, y), f(y, y, \ldots, y)). \end{aligned}$$

By (2.7), it follows that

$$\begin{aligned} d(F(x), F(y)) &\leq a \max\{d(x, f(x, \ldots, x)), d(y, f(y, \ldots, y))\} \\ &\quad + a \max\{d(x, f(x, \ldots, x)), d(y, f(y, \ldots, y))\} \\ &\quad + \ldots + a \max\{d(x, f(x, \ldots, x)), d(y, f(y, \ldots, y))\} \\ &= ak \max\{d(x, f(x, \ldots, x)), d(y, f(y, \ldots, y))\} \\ &\leq ak[d(x, F(x)) + d(y, F(y))] \end{aligned}$$

and we have

$$d(F(x), F(y)) \leq \lambda\, [d(x, F(x)) + d(y, F(y))], \tag{2.8}$$

where $\lambda = ak \in [0, \frac{1}{2})$. So F is a Kannan operator [12]. According to Theorem 1 of [12], there exists a unique $u \in X$ such that

$$u = Fu = f(u, \ldots, u).$$

Thus (i) is proved .

Now, for any arbitrary points $x_1, \ldots, x_k \in X$, we shall prove the convergence of the sequence $\{x_n\}$ defined by (1.1) to u, the unique fixed point of f. For all $n \geq k+1$, we have

$$x_n = f(x_{n-k}, \ldots, x_{n-1}).$$

As we already know that f has a unique fixed point $u \in X$, we may write

$$\begin{aligned} d(x_{n+1}, u) &= d(f(x_{n-k+1}, x_{n-k+2}, \ldots, x_n), f(u, u, \ldots, u)) \\ &\leq d(f(x_{n-k+1}, \ldots, x_n), f(x_{n-k+2}, \ldots, x_n, u)) \\ &\quad + d(f(x_{n-k+2}, \ldots, x_n, u), f(x_{n-k+3}, \ldots, x_n, u, u)) \\ &\quad + \ldots + d(f(x_n, u, \ldots, u), f(u, u, \ldots, u)). \end{aligned}$$

This implies from (2.7) that

$$\begin{aligned} d(x_{n+1}, u) &\leq a \max\{d(x_{n-k+1}, F(x_{n-k+1})), \ldots, d(x_n, F(x_n)), d(u, Fu)\} \\ &\quad + a \max\{d(x_{n-k+2}, F(x_{n-k+2})), \ldots, d(x_n, F(x_n)), d(u, Fu), d(u, Fu)\} \\ &\quad + \ldots + a \max\{d(x_n, F(x_n)), d(u, F(u)), \ldots, d(u, F(u))\}. \end{aligned}$$

Since $u = F(u)$, we obtain

$$\begin{aligned} d(x_{n+1}, u) &\leq a \max\{d(x_{n-k+1}, F(x_{n-k+1})), \ldots, d(x_n, F(x_n))\} \\ &\quad + a \max\{d(x_{n-k+2}, F(x_{n-k+2})), \ldots, d(x_n, F(x_n))\} \\ &\quad + \ldots + a d(x_n, F(x_n)). \end{aligned} \tag{2.9}$$

On the other hand, for all $j \in \mathbb{N}$, we have

$$d(x_j, F(x_j)) \leq d(x_j, u) + d(u, F(x_j)). \tag{2.10}$$

By (2.8), we have

$$\begin{aligned} d(u, F(x_j)) &= d(F(u), F(x_j)) \\ &\leq \lambda\, [d(u, F(u)) + d(x_j, F(x_j))] \\ &= \lambda\, d(x_j, F(x_j)). \end{aligned}$$

Thus (2.10) becomes

$$d(x_j, F(x_j)) \leq d(x_j, u) + \lambda\, d(x_j, F(x_j)),$$

which yields

$$d(x_j, F(x_j)) \leq \frac{1}{1-\lambda}\, d(x_j, u), \text{ for all } j \in \mathbb{N}. \tag{2.11}$$

Using (2.9) and (2.11), we obtain

$$\begin{aligned} d(x_{n+1}, u) &\leq \frac{a}{1-\lambda} \max\{d(x_{n-k+1}, u), \ldots, d(x_n, u)\} \\ &\quad + \frac{a}{1-\lambda} \max\{d(x_{n-k+2}, u), \ldots, d(x_n, u)\} \\ &\quad + \ldots + \frac{a}{1-\lambda} d(x_n, u)) \\ &\leq \frac{ak}{1-\lambda} \max\{d(x_{n-k+1}, u), \ldots, d(x_n, u)\} \end{aligned} \tag{2.12}$$

for all $n \geq k$. Denoting

$$\Delta_n = d(x_n, u), \text{ for all } n \in \mathbb{N}$$

and

$$\alpha = \frac{ak}{1-\lambda},$$

we get

$$\Delta_{n+1} \leq \alpha \max\{\Delta_{n-k+1}, \Delta_{n-k+2}, \ldots, \Delta_n\},$$

for all $n \geq k$. Since, we have $0 \leq \alpha < 1$, follows the similar arguments from Lemma 2 in [18] that there exist $L > 0$ and $\theta \in (0, 1)$ such that $\Delta_n \leq L\theta^n$ for all $n \in \mathbb{N}$, namely such that

$$d(x_n, u) \leq L\theta^n, \text{ for all } n \geq 1.$$

On taking limit as $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} d(x_n, u) = 0$, so the sequence $\{x_n\}$ converges in (X, d) to the unique fixed point of f. Now the proof is complete. □

Corollary 2.6. Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ a given mapping. Suppose that there exists a constant $a \in \mathbb{R}$ with $0 < ak(k+1) < 1$ such that

$$d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1})) \leq a \sum_{i=1}^{k+1} d(x_i, f(x_i, \dots, x_i)), \tag{2.13}$$

for all $(x_1, \dots, x_{k+1}) \in X^{k+1}$. Then,
(i) f has a unique fixed point $u \in X$;
(ii) for any arbitrary points $x_1, \dots, x_k \in X$, the sequence $\{x_n\}$ defined by (1.1) converges to u.

Remark 2.7.
1. Theorem 2.1 extends and generalizes Theorem 1.3 of Ćirić and Prešić [8], and Theorem 1.2 of Prešić [18].
2. If $k = 1$, Theorem 2.1 reduces to the fixed point theorem of Rhoades [19].
3. If $k = 1$, Corollary 2.3 reduces to Theorem 1 of Banach [2].
4. Theorem 2.5 extends the Theorem 1.4 of Păcurar [17].
5. If $k = 1$, Theorem 2.5 reduces to Theorem 1 of Kannan [12].

3. Global Attractivity Results

We investigate the global attractivity of the recursive sequence $\{X_n\} \subset P(N)$ defined by

$$X_{n+k} = Q + \frac{1}{k} \sum_{i=0}^{k-1} A^* \varphi(X_{n+i}) A, \quad n = 1, 2, \dots, \tag{3.1}$$

where $P(N)$ is the set of $N \times N$ Hermitian positive definite matrices, k is a positive integer, Q is an $N \times N$ Hermitian positive semidefinite matrix, A is an $N \times N$ nonsingular matrix, A^* is the conjugate transpose of A and $\varphi : P(N) \to P(N)$.

First we recall some definitions and preliminary results.
Definition 3.1. Let k be a positive integer, M a nonempty set and $f : M^k \to M$. For given $x_1, x_2, \dots, x_k \in M$, consider the recursive sequence $\{x_n\} \subset M$ defined by

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}), \quad n = 1, 2, \dots, \tag{3.2}$$

An equilibrium point $\overline{x}$ of the equation (3.2) is the point that satisfies the condition:

$$\overline{x} = f(\overline{x}, \dots, \overline{x}).$$

Definition 3.2. Let (M, d) be a metric space and $\overline{x}$ an equilibrium point of Eq. (3.2). The equilibrium point $\overline{x}$ is called a global attractor if for all $x_1, x_2, \dots, x_k \in M$, we have $d(x_n, \overline{x}) \to 0$ as $n \to \infty$.

We denote by $P(N)$ (for $N \geq 2$), the open convex cone of all $N \times N$ Hermitian positive definite matrices. We endow $P(N)$ with the Thompson metric defined by

$$A, B \in P(N), \quad d(A, B) = \max\{\ln M(A/B), \ln M(B/A)\},$$

where $M(A/B) = \inf\{\theta > 0 : A \leq \theta B\} = \theta^+(B^{-1/2} A B^{-1/2})$, the maximal eigenvalue of $B^{-1/2} A B^{-1/2}$. Here, $X \leq Y$ means that $Y - X$ is positive semidefinite and $X < Y$ means that $Y - X$ is positive definite. From Nussbaum [15], $P(N)$ is a complete metric space with respect to the Thompson metric d and $d(A, B) = \| \ln(A^{-1/2} B A^{-1/2}) \|$, where $\| \cdot \|$ stands for the spectral norm. The Thompson metric exists on any open normal convex cones of real Banach spaces [15, 25]; in particular, the open convex cone of positive definite operators of a Hilbert space. Now we shortly introduce the elegant properties of the Thompson metric. It is invariant under the matrix inversion and congruence transformations, that is,

$$d(A, B) = d(A^{-1}, B^{-1}) = d(U^* A U, U^* B U), \tag{3.3}$$

for any nonsingular matrix U. The other useful result is the nonpositive curvature property of the Thompson metric

$$d(X^r, Y^r) \leq rd(X, Y), \quad r \in [0, 1]. \tag{3.4}$$

According to (3.3) and (3.4), we have

$$d(U^*X^rU, U^*Y^rU) \leq |r|d(X, Y), \quad r \in [-1, 1]. \tag{3.5}$$

Lemma 3.3. For any $A, B, C, D \in P(N)$,

$$d(A + B, C + D) \leq \max\{d(A, C), d(B, D)\}.$$

Furthermore, for all positive semidefinite A and $B, C \in P(N)$,

$$d(A + B, A + C) \leq d(B, C).$$

Definition 3.4. Let (M, d) be a metric space and $\varphi : M \to M$. We say that φ is α-contraction, if there exists a constant $\alpha \in [0, 1)$ such that

$$d(\varphi(x), \varphi(y)) \leq \alpha d(x, y),$$

for all $x, y \in M$.

Let $\varphi : P(N) \to P(N)$ be an α-contraction with respect to the Thompson metric d. Let Q be an $N \times N$ Hermitian positive semidefinite matrix ($Q \geq 0$) and A an $N \times N$ nonsingular matrix (A^{-1} exists). For a positive integer k, for given $X_1, X_2, \ldots, X_k \in P(N)$, consider the sequence $\{X_n\} \subset P(N)$ defined by (3.1). Our main result in this section is the following.

Theorem 3.5. Eq. (3.1) has a unique equilibrium point $\overline{X} \in P(N)$. Moreover, $\overline{X}$ is global attractor.

Proof. Define the mapping $f : P(N)^k \to P(N)$ by

$$f(U_1, U_2, \ldots, U_k) = Q + \frac{1}{k}[A^*\varphi(U_1)A + A^*\varphi(U_2)A + \ldots + A^*\varphi(U_k)A],$$

for all $U_1, U_2, \ldots, U_k \in P(N)$.

Let $U_1, U_2, \ldots, U_{k+1} \in P(N)$. Using Lemma 3.3, we have

$$\begin{aligned} d(f(U_1, U_2, \ldots, U_k), f(U_2, U_3, \ldots, U_{k+1})) &= d\left(Q + \frac{1}{k}\sum_{i=1}^{k} A^*\varphi(U_i)A, Q + \frac{1}{k}\sum_{j=2}^{k+1} A^*\varphi(U_j)A\right) \\ &\leq d\left(\frac{1}{k}\sum_{i=1}^{k} A^*\varphi(U_i)A, \frac{1}{k}\sum_{j=2}^{k+1} A^*\varphi(U_j)A\right) \\ &= d\left(\sum_{i=1}^{k}\left(\frac{1}{\sqrt{k}}A\right)^* \varphi(U_i)\left(\frac{1}{\sqrt{k}}A\right), \sum_{j=2}^{k+1}\left(\frac{1}{\sqrt{k}}A\right)^* \varphi(U_j)\left(\frac{1}{\sqrt{k}}A\right)\right). \end{aligned}$$

Denote $V = \frac{1}{\sqrt{k}}A$. Then, using again Lemma 3.3, we have

$$\begin{aligned} &d(f(U_1, U_2, \ldots, U_k), f(U_2, U_3, \ldots, U_{k+1}) \\ &\leq d\left(\sum_{i=1}^{k} V^*\varphi(U_i)V, \sum_{j=2}^{k+1} V^*\varphi(U_j)V\right) \\ &= d\left(V^*\varphi(U_1)V + V^*\varphi(U_2)V + \ldots + V^*\varphi(U_k)V, V^*\varphi(U_2)V + V^*\varphi(U_3)V + \ldots + V^*\varphi(U_{k+1})V\right) \\ &\leq \max\left\{d(V^*\varphi(U_1)V, V^*\varphi(U_2)V), d(V^*\varphi(U_2)V, V^*\varphi(U_3)V), \ldots, d(V^*\varphi(U_k)V, V^*\varphi(U_{k+1})V)\right\} \\ &= \max\left\{d(V^*\varphi(U_i)V, V^*\varphi(U_{i+1})V) : i = 1, 2, \ldots, k\right\}. \end{aligned}$$

Since A is nonsingular, the matrix V is also nonsingular. Using property (3.3), for all $i = 1, 2, \dots, k$, we have

$$d(V^*\varphi(U_i)V, V^*\varphi(U_{i+1})V) = d(\varphi(U_i), \varphi(U_{i+1})).$$

But φ is an α-contraction. Then, for all $i = 1, 2, \dots, k$, we have

$$d(V^*\varphi(U_i)V, V^*\varphi(U_{i+1})V) \leq \alpha d(U_i, U_{i+1}).$$

Thus, we have

$$d(f(U_1, U_2, \dots, U_k), f(U_2, U_3, \dots, U_{k+1}) \leq \alpha \max\{d(U_i, U_{i+1}) : \ i = 1, 2, \dots, k\}$$

for all $U_1, U_2, \dots, U_{k+1} \in P(N)$.

Now, Applying Corollary 2.3, we obtain the existence of a global attractor equilibrium point $\overline{X} \in P(N)$.

On the other hand, for $U, W \in P(N)$ such that $U \neq W$, we have

$$\begin{aligned} d(f(U, U, \dots, U), f(W, W, \dots, W)) &= d(Q + A^*\varphi(U)A, Q + A^*\varphi(W)A) \\ &\leq d(A^*\varphi(U)A, A^*\varphi(W)A) \\ &= d(\varphi(U), \varphi(W)) \\ &\leq \alpha d(U, W) \\ &< d(U, W). \end{aligned}$$

Again, applying Corollary 2.3, we obtain the uniqueness of the equilibrium point. □

Now, we present some examples and numerical experiments.

For a positive integer k, consider the sequence $\{X_n\} \subset P(N)$ defined by

$$X_{n+k} = Q + \frac{1}{k}\sum_{i=0}^{k-1} A^*X_{n+i}^{\delta}A, \quad n = 1, 2, \dots \tag{3.6}$$

for given $X_1, X_2, \dots, X_k \in P(N)$, where $|\delta| \in [0, 1)$.

Corollary 3.6. Eq. (3.6) has a unique equilibrium point $\overline{X} \in P(N)$. Moreover, $\overline{X}$ is global attractor.

Proof. Using Properties (3.3) and (3.5), we show easily that $\varphi : P(N) \to P(N)$ defined by

$$\varphi(X) = X^{\delta}, \quad \text{for all } X \in P(N)$$

is $|\delta|$-contraction. Then, the result follows immediately from Theorem 3.5. □

Remark 3.7. The equilibrium point $\overline{X} \in P(N)$ of Eq. (3.6) is the unique positive definite solution to the nonlinear matrix equation

$$\overline{X} = Q + A^*\overline{X}^{\delta}A. \tag{3.7}$$

In the last few years there has been a constantly increasing interest in developing the theory and numerical approaches for positive definite solutions to the nonlinear matrix equation of the form (3.7) (see, for example, [5, 9, 14]).

As an example, we consider for given $X_1, X_2 \in P(N)$, the recursive sequence $\{X_n\} \subset P(N)$ given by

$$X_{n+2} = Q + \frac{1}{2}\left(A^*X_n^{1/2}A + A^*X_{n+1}^{1/2}A\right), \quad n = 1, 2, \dots \tag{3.8}$$

From Corollary 3.6, Eq. (3.8) has a unique equilibrium point $\overline{X} \in P(N)$, that is, the unique positive definite solution to

$$\overline{X} = Q + A^*\overline{X}^{1/2}A.$$

To check our global attractivity result, we give the following numerical experiments.

We take $N = 3$, Q and A are given by

$$Q = \begin{pmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.2 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

For each iteration i, we consider the residual error $E(i)$ given by

$$E(i) = \left\| X_i - (Q + A^* X_i^{1/2} A) \right\|,$$

where $\|\cdot\|$ is the spectral norm. All programs are written in MATLAB version 7.1.

Let us take

$$X_1 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} 5 & 5 & 1 \\ 5 & 11 & 7 \\ 1 & 7 & 17 \end{pmatrix},$$

then after 90 iterations of iterative method (3.8), we get the unique equilibrium point

$$\overline{X} \approx X_{90} = \begin{pmatrix} 438.4 & 429.2 & 429.2 \\ 429.2 & 438.4 & 429.2 \\ 429.2 & 429.2 & 438.4 \end{pmatrix}, \tag{3.9}$$

and its residual error $E(90) = 1.0503e - 013$.

For other initial points

$$X_1 = \begin{pmatrix} 120 & 7 & 7 \\ 7 & 120 & 7 \\ 7 & 7 & 120 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1003 & 3 & 3 \\ 3 & 2003 & 3 \\ 3 & 3 & 3003 \end{pmatrix},$$

after 90 iterations, we get the unique equilibrium point $\overline{X}$ given by (3.9), and its residual error $E(90) = 2.0196e - 013$. □

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